

Solution of the Dirichlet boundary value problem for the Sine-Gordon equation*

Jérôme LEON

*Physique Mathématique et Théorique, CNRS-UMR5825,
Université Montpellier 2, 34095 MONTPELLIER (France)*

Abstract

The sine-Gordon equation in light cone coordinates is solved when Dirichlet conditions on the L-shape boundaries of the strip $\{t \in [0, T]\} \cup \{x \in [0, \infty]\}$ are prescribed in a class of functions that vanish $(\text{mod } 2\pi)$ as $x \rightarrow \infty$ at initial time. The method is based on the inverse spectral transform (IST) for the Schrödinger spectral problem on the semi-line $x > 0$ solved as a Hilbert boundary value problem. Contrarily to what occurs when using the Zakharov-Shabat eigenvalue problem, the spectral transform is regular and in particular the discrete spectrum contains a finite number of eigenvalues (and no accumulation point).

1 Introduction

The sine-Gordon equation in light cone coordinates (dimensionless variables, subscript stands for partial derivative) for the field $\theta(x, t)$,

$$\theta_{xt} + \sin \theta = 0 , \quad (1.1)$$

is one of the most widely studied nonlinear wave equation because of its intrinsic mathematical properties and its wide applicability in physics. The initial value problem on the infinite line $x \in (-\infty, +\infty)$, namely the initial datum $\theta(x, 0)$ together with the asymptotic behaviors $\theta = 2n\pi$ as $x \rightarrow \pm\infty$, has been solved by the *inverse spectral transform* (IST) in [1, 2].

The boundary value problem for (1.1) on the quadrant $\{x > 0, t > 0\}$ has been solved in [3] and reconsidered in [4]. In both approaches it has been shown that a constant boundary datum $\theta(0, t)$ on an initial vacuum $\theta(x, 0) = 0$ produces an infinite series of discrete eigenvalues with accumulation point at the origin. The problem is then that the Jost solutions are not continuous in a neighborhood of the real axis and the spectral theory fails.

*Work supported in part by INTAS 99-1782

We solve this problem by considering an alternative Lax pair for the sine-Gordon equation deduced from the work [5] on two-dimensional extensions of sine-Gordon. The principal spectral operator is the Schrödinger eigenvalue problem with *complex potential* that we reconsider on the half-line $x > 0$ with a given boundary value of the potential in $x = 0$.

The spectral theory developped here follows the approach introduced in [6] for stimulated Raman scattering, reformulated in [4], and later applied to others continuous [7] or discrete [8] coupled wave systems. In the next section the Lax pair is set and the boundary value problem is discussed. Section 3 is devoted to the solution of the *direct spectral problem*, namely to the method of construction of the spectral transform, or *nonlinear Fourier transform*, of the sine-Gordon field.

The time dependence of the spectral transform is then derived in section 4 where it appears to be a nonlinear Riccati type evolution. It is examined for particular boundary data and shown to lead asymptotically to a finite set of discrete eigenvalues (soliton component of the spectrum). In section 5 we solve the *inverse problem* which is the reconstruction of the sine-Gordon field from the spectral transform. This is done by formulating the reconstruction as a Hilbert boundary value problem. In doing this attention must be paid to the asymptotic boundary values in the spectral plane required for the solutions of this Hilbert boundary value problem. Indeed one of the Jost solution has an essential singularity for large real values of the spectral parameter. Still the solution can be obtained by a Cauchy-Green integral equation with a contour of integration that differs from the one used in the infinite line case. The spectral theory of the Schrödinger operator on the half-line can also be developped within the Gel'Fand-Levitan-Marchenko formalism as shown in [9] where the boundary value problem for the Kortevég-de Vries equation is studied.

2 Alternative Lax pair for sine-Gordon

Weakly commuting operators. Let us consider the following pair of scalar operators: the *principal spectral operator*

$$T_1 = \partial_{xx} + k^2 - p , \quad (2.1)$$

where $p(x, t)$ is the *principal potential*, and the *auxiliary spectral operator*

$$T_2 = \partial_{xt} + \alpha\partial_t + \beta , \quad (2.2)$$

where $\alpha(x, t)$ and $\beta(x, t)$ are the *auxiliary potentials*.

Originally introduced for two-dimensional problems [10], the *weak compatibility* of those operators results in requiring that they commute on their common eigenspace. In other words

$$[T_1, T_2] = f T_1 + g T_2 , \quad (2.3)$$

where f and g are two unknown functions. Straightforward examination of highest order derivatives gives

$$f = 0 , \quad g = 2\alpha_x , \quad (2.4)$$

hence we are left with the operatorial equation

$$[T_1, T_2] = 2\alpha_x T_2 . \quad (2.5)$$

This equation explicitly implies the following system

$$p_x = -\alpha_{xx} + 2\alpha\alpha_x , \quad (2.6)$$

$$p_t = -2\beta_x , \quad (2.7)$$

$$p_{xt} + \alpha p_t = -\beta_{xx} + 2\alpha_x\beta , \quad (2.8)$$

in which two equations allow to eliminate two fields and the remaining equation is the nonlinear evolution.

The sine-Gordon equation. Assuming for all t the asymptotic behaviors (we will prove later that these behaviors *result* from the properties of the initial datum)

$$x \rightarrow \infty : \{p \rightarrow 0 , \alpha \rightarrow 0 , \alpha_x \rightarrow 0 , \beta_x \rightarrow 0\} , \quad (2.9)$$

two of the above equation can be integrated and give

$$\beta_x = -2\alpha\beta , \quad p = \alpha^2 - \alpha_x , \quad (2.10)$$

while the remaining equation reads as the evolution

$$p_t = -2\beta_x . \quad (2.11)$$

To eliminate two fields we define the new variable $\theta(x, t)$ by

$$\beta = \frac{1}{4}e^{i\theta} , \quad (2.12)$$

which from (2.9) will be assumed to obey

$$\theta_x \xrightarrow{x \rightarrow \infty} 0 . \quad (2.13)$$

The relation (2.10) then gives p and α in terms of θ as

$$\alpha = -\frac{i}{2}\theta_x , \quad p = \frac{i}{2}\theta_{xx} - \frac{1}{4}\theta_x^2 , \quad (2.14)$$

where one can check that the behavior (2.13) ensures the asymptotic expressions (2.9).

The evolution $p_t = -2\beta_x$ can then be factorized as

$$\left\{ \frac{\partial}{\partial x} + i\theta_x \right\} (\theta_{xt} + \sin \theta) = 0 . \quad (2.15)$$

This can be integrated once and the integration constant can be evaluated by reading the evolution as $x \rightarrow \infty$ where $\theta \rightarrow \lambda$. The results is the following complex extension of the sine-Gordon equation

$$\theta_{xt} + e^{i\lambda} \sin(\theta - \lambda) = 0 , \quad \lambda = \lim_{x \rightarrow \infty} \theta \quad (2.16)$$

associated to the asymptotic behavior (2.13). Note that in general the parameter λ is a function of time t .

In the case of real valued θ , and hence real valued λ , the above evolution implies $\lambda = 0$ and hence reduces to

$$\theta \in \mathbb{R} \Rightarrow \theta_{xt} + \sin \theta = 0 , \quad \theta \xrightarrow[x \rightarrow \infty]{} 0 \mod 2\pi . \quad (2.17)$$

Thus the standard sine-Gordon equation results as the *weak compatibility* of operators T_1 and T_2 for real valued field θ .

It is important to remark here that the imposed vanishing asymptotic behavior of the derivative θ_x for all t as given in (2.13) actually results from the above assumed asymptotic behavior of θ as soon as θ_x vanishes at infinity at initial time. Indeed, by integration of the sine-Gordon equation we can write

$$\theta_x(x, t) = \theta_x(x, 0) - \int_0^t d\tau \sin \theta(x, \tau) . \quad (2.18)$$

This expression serves actually as a means to go from Dirichlet to Neuman conditions on the boundaries $x = 0$ and $x = \infty$. It is also useful to prove that the assumed vanishing asymptotic behavior of the principal potential $p(x, t)$ results from the asymptotic value of $\theta(x, t)$. Indeed, upon derivation of (2.18) it is easy to obtain that if the initial datum is such that $\theta_x(x, 0)$ and $\theta_{xx}(x, 0)$ vanish as $x \rightarrow \infty$, then θ_{xx} vanishes for all time as $x \rightarrow \infty$.

Note also that the expression (2.16) of the sine-Gordon equation for complex variable can be put in the form (2.17) by the following change of variables

$$\varphi = \theta - \lambda , \quad \tau = \int_0^t dt' e^{-i\lambda(t')} , \quad (2.19)$$

which gives

$$\varphi_{x\tau} + \sin \varphi = 0 , \quad \varphi \xrightarrow[x \rightarrow \infty]{} 0 \mod 2\pi . \quad (2.20)$$

We have assumed hereafter the general situation where the constant λ is allowed to depend on time t .

Dirichlet boundary data. We consider then the sine-Gordon equation (1.1), a priori for a complex field θ , on the strip

$$\{x \in [0, \infty]\} \cup \{t \in [0, T]\} , \quad (2.21)$$

for any value of T . The Dirichlet conditions are the data of the function $\theta(x, t)$ on the L-shape boundaries of this strip, namely

$$t = 0 : \theta(x, 0) = \tilde{\theta}(x) , \quad x = 0 : \theta(0, t) = \theta_0(t) . \quad (2.22)$$

The auxiliary spectral operator (2.2) explicitly contains the variable $\alpha(x, t)$ which has then to be evaluated in $x = 0$ explicitly in terms of the Dirichlet data (2.22). By expressing (2.18) in $x = 0$ we have

$$\theta_x(0, t) = \tilde{\theta}_x(0) - \int_0^t d\tau \sin \theta_0(\tau) , \quad (2.23)$$

which from (2.14) gives

$$\alpha_0(t) = \frac{i}{2} \int_0^t d\tau \sin \theta_0(\tau) - \frac{i}{2} \tilde{\theta}_x(0) , \quad (2.24)$$

where $\alpha_0 = \alpha(0, t)$ is then completely determined from (2.22). Similarly we have to evaluate $p_0(t) = p(0, t)$ occurring in the principal Lax operator. By derivation of (2.18) we obtain

$$p_0(t) = \frac{i}{2} \tilde{\theta}_{xx}(0) - \frac{i}{2} \int_0^t d\tau \theta_x(0, \tau) \cos \theta_0(\tau) - \frac{1}{4} \theta_x^2(0, t) , \quad (2.25)$$

where $\theta_x(0, t)$ is explicitly given in (2.23). We have then obtained that the data of $\theta_0(t)$ and $\tilde{\theta}(x)$ determine completely the boundary values in $x = 0$ of the potentials of the fundamental operators T_1 and T_2 .

For the other boundary $x = \infty$ we first select a class of functions for the initial datum $\tilde{\theta}(x)$. As usual for implementation of the spectral transform, let us assume that $\tilde{\theta}(x)$ belongs to the Schwartz space \mathcal{S} (bounded functions that vanish with all their derivatives faster than any power of $1/x$). Although this condition is rather stringent, it fits well a generic physical situation, in particular the case of an initial *vacuum*. It is then easy to obtain from (2.18) the property

$$\left\{ \tilde{\theta}(x) \in \mathcal{S}, \quad \theta(\infty, t) = 0 \pmod{2\pi} \right\} \Rightarrow \theta_x(\infty, t) = 0 . \quad (2.26)$$

This can be worked again for the second derivative and finally the potentials $p(x, t)$ and $\alpha(x, t)$ do vanish as $x \rightarrow \infty$ and β goes asymptotically to $1/4$.

3 Spectral problem

The Schrödinger spectral problem on the half-line $T_1\psi = 0$ with complex potential, namely

$$x > 0, \quad \psi_{xx} + (k^2 - p)\psi = 0, \quad p(0) = p_0, \quad (3.1)$$

is solved now and the inverse problem will be tackled by means of a Hilbert boundary value problem. A partial solution has already been given in [7], in the case of real valued potential. We rework everything here with slightly different definitions and in particular give the complete solution of the Hilbert boundary value problem. Note that this spectral problem has also been considered in [11].

Basic solutions. We redefine the spectral equation as

$$\psi = \phi e^{ikx}, \quad \phi_{xx} + 2ik\phi_x - p\phi = 0, \quad (3.2)$$

and select two basic solutions ϕ^+ and ϕ^- by

$$\phi^-(k, x) = 1 + \frac{1}{2ik} \int_0^x dy \phi^-(k, y)p(y)[1 - e^{-2ik(x-y)}], \quad (3.3)$$

$$\begin{aligned} \phi^+(k, x) = 1 + \frac{1}{2ik} \int_0^x dy \phi^+(k, y)p(y) \\ + \frac{1}{2ik} \int_x^\infty dy \phi^+(k, y)p(y)e^{-2ik(x-y)}. \end{aligned} \quad (3.4)$$

where conditions on $p(x)$ are assumed (e.g. $p(x)$ belongs to the Schwartz space) that ensure uniqueness of the solution of the above integral equations. It is useful to compute the derivatives

$$\phi_x^-(k, x) = \int_0^x dy \phi^-(k, y)p(y)e^{-2ik(x-y)}, \quad (3.5)$$

$$\phi_x^+(k, x) = - \int_x^\infty dy \phi^+(k, y)p(y)e^{-2ik(x-y)}. \quad (3.6)$$

An important property for the following is that another set of basic solutions is constituted by $\phi^\pm(-k, x)e^{-2ikx}$. Another important property is the fact that the wronskian of two solutions of (3.1) is a constant, which means for any two solutions ϕ_1 and ϕ_2 of (3.2)

$$W(\phi_1, \phi_2) = \phi_{1,x}\phi_2 - \phi_1\phi_{2,x} = C e^{-2ikx}, \quad (3.7)$$

where C is a constant.

Spectral transform. The solution ϕ^+ is called the *physical solution* and serves to define the scattering coefficients as follows. The boundary value in $x = 0$

$$x = 0 : \phi^+ = 1 + \rho(k), \quad (3.8)$$

defines the *reflection coefficient* $\rho(k)$ as

$$\rho(k) = \frac{1}{2ik} \int_0^\infty dy p(y) \phi^+(k, y) e^{2iky} , \quad (3.9)$$

and consequently (3.6) gives

$$x = 0 : \phi_x^+ = -2ik\rho(k) . \quad (3.10)$$

The boundary value in $x = \infty$

$$x \rightarrow \infty : \phi^+ \rightarrow \tau(k) , \quad (3.11)$$

defines the *transmission coefficient* $\tau(k)$ as

$$\tau(k) = 1 + \frac{1}{2ik} \int_0^\infty dy p(y) \phi^+(k, y) , \quad (3.12)$$

and we have also

$$x \rightarrow \infty : \phi_x^+ \rightarrow 0 . \quad (3.13)$$

The wronskian relation (3.7) applied to the solutions $\phi^+(k)$ and $\phi^+(-k)e^{-2ikx}$ at $x = 0$ and $x = \infty$ successively gives

$$\rho(k)\rho(-k) + \tau(k)\tau(-k) = 1 , \quad (3.14)$$

called the *unitarity relation*.

From its very definition, the spectral transform $\rho(k)$ seems to be ill defined in $k = 0$. Actually we recall in the appendix A that

$$k = 0 : \rho = -1 , \quad \tau = 0 , \quad (3.15)$$

which is well known in the infinite line case [12].

Hilbert boundary value problem The solution ϕ^- is called the *intermediate solution* and serves to complete the basis. We can for instance express $\phi^+(k)$ in terms of the basis $\phi^-(k)$ and $\phi^-(-k)e^{-2ikx}$, the coefficients being computed by comparing the values in $x = 0$ of both the functions and their derivatives. We obtain

$$\phi^+(k, x) - \phi^-(k, x) = \rho(k)e^{-2ikx}\phi^-(-k, x) , \quad (3.16)$$

which constitutes a Hilbert problem of the real k -axis, we shall come to that argument later. For the moment we express the above relation by changing k to $-k$ and solve the resulting linear system for the function $\phi^-(k)$ to obtain

$$\phi^-(k, x) = \frac{1}{1 - \rho(k)\rho(-k)} [\phi^+(k, x) - \rho(k)e^{-2ikx}\phi^+(-k, x)] . \quad (3.17)$$

Using the definitions of ϕ^- and ϕ_x^- and the above relation we can compute the different boundary values and get

$$x = 0 \quad : \phi^- = 1 , \quad \phi_x^- = 0 , \quad (3.18)$$

$$x = \infty \quad : \phi^- \sim \frac{1}{\tau(-k)} - \frac{\rho(k)}{\tau(k)} e^{-2ikx} , \quad \phi_x^- \sim 2ik \frac{\rho(k)}{\tau(k)} e^{-2ikx} , \quad (3.19)$$

where we have also used the unitarity relation (3.14).

4 Solution of sine-Gordon

We have constructed in the preceding section the spectral transform $\rho(k)$ of the principal potential $p(x)$. We shall prove in section 5 that the datum of $\rho(k)$ in the upper half plane completely determines the potential $p(x)$, and we shall give the tools to reconstruct this potential. But we consider first the question of the *time dependence* of the spectral transform when the potential depends on time according to the evolutions (2.10), reducing to sine-Gordon in the real valued variable θ .

Evolution of the spectral transform. From the weak compatibility commutation (2.5) of operators T_1 and T_2 , the selected eigenfunctions ψ^\pm are not eigenfunctions of T_2 and our purpose here is to determine their proper time evolution. Equation (2.5) implies the property

$$T_1\psi = 0 \Rightarrow (T_1 - 2\alpha_x)T_2\psi = 0 , \quad (4.1)$$

and thus using the remarkable identity

$$T_1\psi = 0 \Rightarrow (T_1 - 2\alpha_x)(\psi_x + \alpha\psi) = 0 , \quad (4.2)$$

which simply results from the structure of T_1 in terms of the potential α , we may write

$$T_2\psi^+ = \Omega (\psi_x^+ + \alpha\psi^+) . \quad (4.3)$$

Hereabove, the x -independent coefficient $\Omega(k, t)$ is called the *generalized dispersion relation*. Note that one should actually expand $T_2\psi^+$ on a basis, say $\psi_x^+(k) + \alpha\psi^+(k)$ and $\psi_x^+(-k) + \alpha\psi^+(-k)$, but examination of the asymptotic behaviors as $x \rightarrow \infty$ readily reduces the expansion to the form (4.3) and gives the evolution

$$\tau_t = \left[\Omega - \frac{1}{4ik} \right] \tau , \quad (4.4)$$

of the transmission coefficient.

The value of the dispersion relation Ω together with the time evolution of ρ are obtained by expressing equation (4.3) and its x -derivative in $x = 0$. We obtain

$$(\alpha_0 - ik)\rho_t + \beta_0(1 + \rho) = \Omega[\alpha_0(1 + \rho) + ik(1 - \rho)] , \quad (4.5)$$

$$\begin{aligned} (\alpha_0^2 - ik\alpha_0 - k^2)\rho_t + 2\alpha_0\beta_0(1 + \rho) + ik\beta_0(1 - \rho) &= \\ \Omega[(\alpha_0^2 - k^2)(1 + \rho) + ik\alpha_0(1 - \rho)] . \end{aligned} \quad (4.6)$$

where we have used the expressions $p = \alpha^2 - \alpha_x$ and $\beta_x = -2\alpha\beta$ and the equation $T_1\psi = 0$. This is a system of equations for the unknowns $\{\rho_t, \Omega\}$ in terms of the boundary data α_0 and β_0 and of ρ . Note that it is compatible with (see appendix A)

$$k = 0 \Rightarrow \{\rho = -1, \tau = 0\} . \quad (4.7)$$

Multiplying the first equation by α_0 and subtracting with the second gives

$$\rho_t = (1 + \rho) \left[\Omega + \beta_0 \frac{\alpha_0}{k^2} \right] - \frac{\beta_0}{ik}(1 - \rho) . \quad (4.8)$$

Replacing in any of the preceding equations, we obtain the dispersion relation which after some algebra can be written

$$\Omega = \rho \frac{\alpha_0\beta_0}{2ik^3}(\alpha_0 - 2ik) + \frac{\beta_0}{2ik^3}(\alpha_0^2 + 2k^2) . \quad (4.9)$$

Thus the evolution of the reflection coefficient eventually reads as the Riccati differential equation

$$\rho_t = a\rho^2 + b\rho + c , \quad (4.10)$$

with entries which depend on the boundary data by

$$a = \frac{\alpha_0\beta_0}{2ik^3}(\alpha_0 - 2ik) , \quad b = \frac{\beta_0}{ik^3}(\alpha_0^2 + 2k^2) , \quad c = \frac{\alpha_0\beta_0}{2ik^3}(\alpha_0 + 2ik) . \quad (4.11)$$

Soliton generation. To illustrate the above formalism we consider the case of a piecewise constant boundary datum, i.e.

$$t = 0 : \tilde{\theta}(x) = 0 , \quad t < t_1 : \theta_0(t) = 0 , \quad t > t_1 : \theta_0(t) = \varphi . \quad (4.12)$$

For such a simple case, the Zakharov-Shabat spectral transform has been shown to generate an infinite number of discrete eigenvalues with $k = 0$ as an accumulation point [4].

Here we have from (2.24)

$$\begin{aligned} t < t_1 : \alpha_0(t) &= 0, \quad \beta_0 = \frac{1}{4} , \\ t > t_1 : \alpha_0(t) &= \frac{i}{2}(t - t_1)\sin\varphi, \quad \beta_0 = \frac{1}{4}e^{i\varphi} . \end{aligned} \quad (4.13)$$

For such expressions, the time-evolution (4.10), with vanishing initial condition $\rho(k, 0) = 0$, cannot be explicitly solved. However we can evaluate the large time asymptotic behavior as indeed

$$\begin{aligned}\frac{b}{a} &= 2 \frac{\alpha_0^2 + 2k^2}{\alpha_0(\alpha_0 - 2ik)} \xrightarrow[t \rightarrow \infty]{} 2, \\ \frac{c}{a} &= \frac{\alpha_0 + 2ik}{\alpha_0 - 2ik} \xrightarrow[t \rightarrow \infty]{} 1.\end{aligned}\quad (4.14)$$

The resulting asymptotic evolution then reads

$$\rho_t \sim a(\rho + 1)^2, \quad (4.15)$$

which is easily solved to eventually get for large time t

$$\rho(k, t) \sim \frac{-e^{i\varphi} \sin \varphi [\frac{1}{3}(t - t_1) \sin \varphi - k]}{32ik^3 + e^{i\varphi} \sin \varphi [\frac{1}{3}(t - t_1) \sin \varphi - k]} \frac{(t - t_1)^2}{(t - t_1)^2}. \quad (4.16)$$

This function has 3 simple poles $k_n(t)$ given by

$$k_n(t) = (t - t_1)Z_n, \quad 32iZ_n^3 - 2e^{i\varphi} \sin \varphi Z_n + \frac{1}{3}e^{i\varphi} \sin^2 \varphi = 0. \quad (4.17)$$

For different values of φ , one or two of these poles are in the upper half plane corresponding to a solution with a one-soliton or a two-soliton component.

Note that the Riccati time evolution (4.10) is not in general explicitly solvable. Indeed, simple explicit solution could only be obtained for b/a and c/a time independent which would immediately lead to $\theta_0(t) = 0$. Such does not occur in the Zakharov-Shabat spectral transform of sine-Gordon where piecewise constant boundary data in $x = 0$ produce an explicitly solvable evolution [4]. Then it has been shown that the spectral transform contains an infinite series of poles, accordingly with [3], see also [13]. The problem is that these poles possess an accumulation point (the origin $k = 0$) and hence some of the basic solutions are not continuous in the vicinity of the origin. Using the present spectral transform based on the Schrödinger spectral problem removes this problem and thus provides the correct solution to the Dirichlet boundary value problem for sine-Gordon in light-cone coordinates.

5 Inverse problem

Solving the inverse problem will consist in computing the potential $p(x)$ from the datum of $\rho(k)$. This is done by solving first the Hilbert boundary value problem (3.16) which actually requires the knowledge of the analytical properties of the functions $\phi^\pm(k)$ together with their behaviors at large k .

Analytical properties. The Volterra integral equation (3.3) allows to deduce that ϕ^- is an entire function of k bounded in the lower half plane. The function ϕ^+ is not analytic but the function $\tilde{\phi}^+ = \phi^+/\tau$ solves the Volterra integral equation

$$\tilde{\phi}^+ = 1 - \frac{1}{2ik} \int_x^\infty dy \tilde{\phi}^+(k, y)p(y) \left[1 - e^{-2ik(x-y)} \right]. \quad (5.1)$$

Hence ϕ^+/τ is analytic in the upper half-plane and thus, from definition (3.12), $1/\tau$ is also analytic. Consequently ϕ^+ has a pole where $1/\tau$ has a zero, these are the *discrete eigenvalues* of the spectral transform named k_n .

The definition (3.9) of the reflection coefficient shows that ρ can be extended to the upper half plane (unlike in the infinite line case where the integral would run on the whole real axis) where it possess the discrete eigenvalues as poles. The integral equation (3.4) can be rewritten as

$$\phi^+ = 1 + \rho e^{-2ikx} + \frac{1}{2ik} \int_0^x dy \phi^+(k, y)p(y)[1 - e^{-2ik(x-y)}]. \quad (5.2)$$

By taking the residue of the above in k_n we get

$$e^{2ik_n x} \text{Res}_{k_n}(\phi^+) = \rho_n - \frac{1}{2ik_n} \int_0^x [e^{2ik_n y} \text{Res}_{k_n}(\phi^+)] p(y) [1 - e^{2ik_n(x-y)}], \quad (5.3)$$

where obviously

$$\rho_n = \text{Res}_{k_n}(\rho). \quad (5.4)$$

By comparaison of the above integral equation with the one for the function $\phi^-(-k, x)$ read in $k = k_n$ we immediately conclude that

$$e^{2ik_n x} \text{Res}_{k_n}(\phi^+(k, x)) = \rho_n \phi^-(-k_n, x). \quad (5.5)$$

This closes the analytical properties needed to the solution of the Hilbert boundary value problem.

Asymptotic behaviors. It is necessary now to determine the large k behaviors of the solutions ϕ^\pm . By repeated integration by part, the solutions obey

$$\phi^+(k, x) \sim 1 + \frac{1}{2ik} \phi^{(1)} + \frac{1}{(2ik)^2} \phi^{(2)} + \dots \quad (5.6)$$

$$\phi^-(k, x) \sim 1 + \frac{1}{2ik} \phi^{(1)} + \frac{1}{(2ik)^2} [\phi^{(2)} + \varphi^{(2)} e^{-2ikx}] + \dots \quad (5.7)$$

with the following coefficients

$$\phi^{(1)} = \int_0^x dy p(y), \quad (5.8)$$

$$\phi^{(2)} = -p(x) + \int_0^x dy p(y) \int_0^y dz p(z), \quad (5.9)$$

$$\varphi^{(2)} = p_0. \quad (5.10)$$

The solution of the Hilbert problem will have to verify those behaviors, with in particular the essential singularity for ϕ^- on the real axis.

The relation (5.8) furnishes the means to obtain the potential p from the solution ϕ^+ namely

$$p(x) = \partial_x \phi^{(1)}(x) , \quad (5.11)$$

where $\phi^{(1)}$ is the coefficient of $1/2ik$ in the large k expansion of $\phi^+(k, x)$. Note finally that the spectral transform $\rho(k)$ obeys in the upper half-plane

$$\rho(k) = -\frac{p_0}{(2ik)^2} + \mathcal{O}(1/k^3) , \quad (5.12)$$

obtained again by integration by part on the definition (3.9) with help of the behavior (5.6). Note that this is consistent with the boundary value of ϕ^+ in $x = 0$ given in (3.8) and the large k behavior (5.6). Indeed

$$\begin{aligned} \phi^+ &\xrightarrow{x \rightarrow 0} 1 + \rho \xrightarrow{k \rightarrow \infty} 1 - \frac{p_0}{(2ik)^2} + \dots \\ \phi^+ &\xrightarrow{k \rightarrow \infty} 1 + \frac{1}{2ik} \phi^{(1)} + \frac{1}{(2ik)^2} \phi^{(2)} + \dots \xrightarrow{x \rightarrow 0} 1 - \frac{p_0}{(2ik)^2} + \dots \end{aligned}$$

This works as well for ϕ^- , hence in summary, the limits $k \rightarrow \infty$ and $x \rightarrow 0$ commute.

Solution of the Hilbert problem. The solution ϕ^\pm of the Hilbert boundary value problem (3.16) which verifies the preceding analytical properties and large k behaviors is given from the data of $\rho(k)$ in $\text{Im}(k) \geq 0$ (including eigenvalues k_n in the upper half-plane and related residues ρ_n) by the Cauchy-Green *integral equation*

$$\begin{aligned} \phi^-(k, x) &= 1 + \frac{1}{2i\pi} \int_{-\infty+i0}^{+\infty+i0} \frac{d\lambda}{\lambda - k} \phi^-(-\lambda, x) \rho(\lambda) e^{-2i\lambda x} \\ &\quad + \sum_{n=1}^N \frac{\rho_n e^{-2ik_n x}}{k_n - k} \phi^-(-k_n, x) , \quad \text{Im}(k) \leq 0 , \end{aligned} \quad (5.13)$$

and the *explicit formula*

$$\begin{aligned} \phi^+(k, x) &= 1 + \frac{1}{2i\pi} \int_{-\infty-i0}^{+\infty-i0} \frac{d\lambda}{\lambda - k} \phi^-(-\lambda, x) \rho(\lambda) e^{-2i\lambda x} \\ &\quad + \sum_{n=1}^N \frac{\rho_n e^{-2ik_n x}}{k_n - k} \phi^-(-k_n, x) , \quad \text{Im}(k) \geq 0 . \end{aligned} \quad (5.14)$$

As shown for the first time in [4], the choice of the particular contour in the complex plane for the above expressions is essential.

Indeed, while the factor $e^{-2i\lambda x}$ produces a bounded kernel for $x > 0$ in the lower half λ -plane, it does not so in the upper half λ -plane, that is for the expression of ϕ^- . Assuming no pole for ρ on the real axis, we can use contour integration on (5.13) to write

$$\begin{aligned}\phi^-(k, x) = 1 + \frac{1}{2i\pi} \int_{-\infty-i0}^{+\infty-i0} \frac{d\lambda}{\lambda - k} \phi^-(-\lambda, x) \rho(\lambda) e^{-2i\lambda x} \\ - \phi^-(-k, x) \rho(k) e^{-2ikx} + \sum_{n=1}^N \frac{\rho_n e^{-2ik_n x}}{k_n - k} \phi^-(-k_n, x).\end{aligned}\quad (5.15)$$

The first consequence is that indeed these solution obey the Hilbert relation (3.16). The second consequence is that they do verify the asymptotic behaviors (5.6) and (5.7) obtained here simply by expanding the ratio $1/(\lambda - k)$ in expressions (5.14) and (5.15) where kernels are bounded, unlike in (5.13).

The analytical properties naturally follow the expressions (5.14) and (5.13). Indeed, as $\text{Im}(k_n) > 0$, the analyticity of the solution $\phi^-(k, x)$ of (5.13) simply result from the property (continuity, boundedness) of $\rho(k)$ on the real axis. Then the solution $\phi^+(k, x)$, given by the explicit expression (5.14), appears as the sum of an analytic function and the contribution of its simple poles k_n . Finally the relation (5.5) between the residue of ϕ^+ in k_n and ρ_n readily follows from expression (5.14).

Reconstruction of $\theta(x, t)$. Once the spectral transform $\rho(k, t)$ has been obtained by solving the Riccati time evolution (4.10) for the initial datum $\rho(k, 0)$, the solution of sine-Gordon $\theta(x, t)$ is built out of the integral equation (5.13) as follows. The relation (5.11) gives the value of $p(x, t)$ from the coefficient $\phi^{(1)}(x, t)$ of $1/2ik$ in the large k expansion of $\phi^+(k, x, t)$, namely

$$\begin{aligned}\phi^{(1)}(x, t) = -\frac{1}{\pi} \int_{-\infty-i0}^{+\infty-i0} d\lambda \phi^-(-\lambda, x, t) \rho(\lambda, t) e^{-2i\lambda x} \\ - 2i \sum_{n=1}^N \phi^-(-k_n, x, t) \rho_n e^{-2ik_n x}.\end{aligned}\quad (5.16)$$

This expression shows the two building components of the solution: the radiation component represented by the continuous integral and the soliton component represented by the discrete sum. Note that the number N and positions k_n of the solitons depend on time as allowed by the Riccati time evolution (4.10).

The simplest way to obtain now the solution $\theta(x, t)$ is to use the relations

$$p_t = -2\beta_x, \quad p = \phi_x^{(1)}, \quad (5.17)$$

easily integrated with the values in $x = 0$ of β . We eventually arrive at

$$e^{i\theta(x, t)} = e^{i\theta_0(t)} - 2\partial_t \phi^{(1)}(x, t). \quad (5.18)$$

This provides then the solution $\theta(x, t)$ from the boundary value $\theta_0(t)$ and the solution $\phi^-(k, x, t)$ of the integral equation (5.13) completely determined by $\rho(k, t)$.

6 Conclusion

The solution of the Dirichlet boundary value problem for the sine-Gordon equation has been shown to be completely characterized by the time-evolution (4.10) of the spectral transform $\rho(k, t)$ evolving explicitly the boundary datum $\theta(0, t)$, with an initial datum $\rho(k, 0)$ determined from the initial condition $\theta(x, 0)$.

This solution has many important aspects. First the solution is *stable* for initial conditions in the Schwartz space, much like what occurs for the infinite line case. Second, the time evolution of the spectral transform is *regular* in the sense that it does not produce accumulation of poles. Last but not least this approach allows to treat the *finite line* case as indeed it is an open-end boundary value problem. This property was already remarked in [6] and later discussed in [8], and mainly results from the fact that the solution is completely determined by the integral equation (5.13) where the x -dependence is *parametric*.

In order now to describe in more details interesting generic behaviors of the solution of sine-Gordon, we would need to study the Riccati time evolution (4.10) where the coefficients are time-dependent. This aspect of the work is reported to further studies as it would require use of numerical simulations.

A Behaviors at the origin

The definition (3.4) of the solution $\phi^+(k, x)$ and the Hilbert relation (3.16) written both in $x = 0$ lead to the system (in $k = 0$ we note $\phi(0, x) = \phi_0(x)$ and $\rho(0) = \rho_0$)

$$\phi_0^+(x) = 1 + \rho_0 - \int_x^\infty dy(x-y)\phi_0^+(y)p(y) , \quad (\text{A.1})$$

$$\phi_0^+(x) = [1 + \rho_0]\phi_0^-(x) , \quad (\text{A.2})$$

while ϕ_0^- solves

$$\phi_0^-(x) = 1 + \int_0^x dy(x-y)\phi_0^-(y)p(y) . \quad (\text{A.3})$$

As the above Volterra integral equation has a unique solution, ϕ_0^- does not vanish and the equation (A.1) can be written by using (A.2) as

$$\phi_0^+(x) [1 - 1/\phi_0^-(x)] = - \int_x^\infty dy(x-y)\phi_0^+(y)p(y) . \quad (\text{A.4})$$

This has the unique vanishing solution and consequently we have proved that

$$\phi_0^+(x) = 0 , \quad \rho_0 = -1 . \quad (\text{A.5})$$

Finally the unitarity relation (3.14) leads immediately $\tau(0) = 0$ as expected.

B Reduction

We determine here the symmetry properties of the basic solutions and the spectral transform resulting from the constraint $\theta \in \mathbb{R}$. First we note that

$$\theta \in \mathbb{R} \Rightarrow T_1^* = T_1 - 2\alpha_x , \quad (\text{B.1})$$

where for any function $f(k)$ of the complex variable k , we define

$$f^* = \overline{f(\bar{k})} . \quad (\text{B.2})$$

Second the operator T_1 has the property (4.2) and consequently when θ is real valued (or equivalently when α is purely imaginary), the eigenfunctions ψ^* and $\psi_x + \alpha\psi$ are linearly dependent, namely

$$(\partial_x + \alpha)\psi^+(k) = A\bar{\psi}^+(\bar{k}) + B\bar{\psi}^+(-\bar{k}) , \quad (\text{B.3})$$

$$(\partial_x + \alpha)\psi^-(k) = C\bar{\psi}^+(\bar{k}) + D\bar{\psi}^+(-\bar{k}) . \quad (\text{B.4})$$

The coefficients hereabove are computed by expressing these expansions in $x = \infty$ with help of (3.11) and (3.19). We obtain using $\alpha(\infty) = 0$

$$A = 0 , \quad B = ik \frac{\tau(k)}{\tau(-\bar{k})} , \quad (\text{B.5})$$

$$C = ik \frac{\rho(k)}{\tau(k)\bar{\tau}(\bar{k})} , \quad D = \frac{ik}{\tau(-k)\bar{\tau}(-\bar{k})} . \quad (\text{B.6})$$

The next step consists in expressing (B.3) and (B.4) in $x = 0$ which results in a set of two equations linking the spectral coefficients and the boundary value α_0 . By eliminating α_0 between these two equations and after some simple algebra using the unitarity relation (3.14) we eventually obtain first

$$\frac{1}{2} \frac{[1 + \bar{\rho}(\bar{k})][1 + \rho(k)]}{\tau(k)\bar{\tau}(\bar{k})} + \frac{1}{2} \frac{[1 + \bar{\rho}(-\bar{k})][1 + \rho(-k)]}{\tau(-k)\bar{\tau}(-\bar{k})} = 1 , \quad (\text{B.7})$$

which constitutes the condition for a spectral transform $\rho(k)$ to ensure a *real valued* reconstructed field $\theta(x)$. The second equation gives a relation between the boundary value α_0 and the spectral transform, namely

$$\alpha_0 = -ik \left[1 - \frac{1 + \bar{\rho}(-\bar{k})}{\tau(-k)\bar{\tau}(-\bar{k})} - \rho(k) \frac{1 + \bar{\rho}(\bar{k})}{\tau(k)\bar{\tau}(\bar{k})} \right] . \quad (\text{B.8})$$

This relation must in particular be valid as $k \rightarrow \infty$ where the behaviors of the spectral coefficients ρ and τ are given by

$$\rho(k) \sim \frac{p_0}{4k^2} + \mathcal{O}\left(\frac{1}{k^3}\right), \quad (\text{B.9})$$

$$\tau(k) \sim 1 + \frac{\gamma}{2ik} + \frac{\gamma^2}{2(2ik)^2} + \mathcal{O}\left(\frac{1}{k^3}\right). \quad (\text{B.10})$$

The behavior of τ is obtained from its definition (3.12) and the large k -behavior (5.6) of the eigenfunction ϕ^+ . By using then the relation (5.8) it is not difficult to obtain

$$\gamma = \alpha_0 + \int_0^\infty dx \alpha^2. \quad (\text{B.11})$$

The reduction constraint $\theta \in \mathbb{R}$ is here equivalent to $\gamma - \bar{\gamma} = 2\alpha_0$. With all this in hands, the above relation (B.8) does hold as $k \rightarrow \infty$ where the zeroth order term in the right hand side is indeed α_0 . One can check then that the next order is a vanishing identity as we have within the reduction

$$p_0 + \bar{p}_0 - 2\alpha_0^2 = 0. \quad (\text{B.12})$$

References

- [1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, Phys Rev Lett **30** (1973) 1262.
- [2] L.A. Takhtajan, L.D. Faddeev, Theor Math Phys **21** (1974) 1046
- [3] A.S. Fokas, Proc Roy Soc Lond A **453** (1997) 1411
- [4] J. Leon, A. Spire, J Phys A **34** (2001) 7359
- [5] M. Boiti, J. Leon, F. Pempinelli, Inv Problems **3** (1987) 37
- [6] J. Leon, A.V. Mikhailov, Phys Lett A **253** (1999) 33
- [7] F-X. Hugot, J. Leon, Inv Problems **15** (1999) 701
- [8] M. Boiti, J. Leon, F. Pempinelli, J Phys A **32** (1999) 927
- [9] P.C. Sabatier, J Math Phys, **41** (2000) 414
- [10] M. Boiti, J. Leon, M. Manna, F. Pempinelli, Inv Problems **2** (1986) 271; *ibid* **3** (1987) 25
- [11] A.S. Fokas, B. Pelloni, Proc Roy Soc Lond A **456** (2000) 805

- [12] K.H. Chadan, P.C. Sabatier, *Inverse problems in quantum scattering theory*, Springer Verlag (New-York 1989). B.M. Levitan, I.S. Sargsjan, *Introduction to spectral theory*, Transl Math Monographs, A.M.S. (Providence 1975).
- [13] B. Pelloni, Comm Appl Anal **6** (2002) 179